Function of Bounded Variation

We shall now discuss the concept of functions of bounded variation which is closely associated to the concept of monotonic functions and has wide application in mathematics. These functions are used in Riemann-Stieltjes integrals and Fourier series.

Let a function f be defined on an interval $[a,b]$ and $P = \{a = x_0, x_1, \ldots\ldots\ldots, x_n = b\}$ be a partition of $[a,b]$. Consider the sum $\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$. The set of these sums is infinite. It changes when we make a refinement in a partition. If this set of sums is bounded above then the function f is said to be a *bounded variation* and the supremum of the set is called the *total variation* of the function f on $[a,b]$, and is denoted by $V(f;a,b)$ or $V_f(a,b)$ and it is also affiliated as $V(f)$ or V_f.

Thus

$$V(f;a,b) = \sup \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

The supremum being taken over all the partition of $[a,b]$.

Hence the function f is said to be of *bounded variation* on $[a,b]$ if, and only, if its total variation is finite i.e. $V(f;a,b) < \infty$.

Note

Since for $x \le c \le y$, we have

$$|f(y) - f(x)| \le |f(y) - f(c)| + |f(c) - f(x)|$$

Therefore the sum $\sum |f(x_i) - f(x_{i-1})|$ can not be decrease (it can, in fact only increase) by the refinement of the partition.

Theorem

A bounded monotonic function is a function of bounded variation.

Proof

Suppose a function f is monotonically increasing on $[a,b]$ and P is any partition of $[a,b]$ then

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1})\right) = f(b) - f(a)$$

$$\therefore\ V(f;a,b) = \sup \sum |f(x_i) - f(x_{i-1})| = f(b) - f(a) \quad \text{(finite)}$$

Hence the function f is of bounded variation on $[a,b]$.

Similarly a monotonically decreasing bounded function is of bounded variation with total variation $= f(a) - f(b)$.

Thus for a bounded monotonic function f

$$V(f) = |f(b) - f(a)|$$ ❑

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Example

A continuous function may not be a function of bounded variation.

e.g. Consider a function f, where

$$f(x)\begin{cases} x\sin\dfrac{\pi}{x} & ; \text{ when } 0<x\le 1 \\ 0 & ; \text{ when } x=0 \end{cases}$$

It is clear that f is continuous on $[0,1]$.

Let us choose the partition $P=\left\{0,\dfrac{2}{2n+1},\dfrac{2}{2n-1},\ldots\ldots,\dfrac{2}{5},\dfrac{2}{3},1\right\}$

Then

$$\sum\left|f(x_i)-f(x_{i-1})\right|=\left|f(1)-f\left(\frac{2}{3}\right)\right|+\left|f\left(\frac{2}{3}\right)-f\left(\frac{2}{5}\right)\right|+\ldots\ldots+\left|f\left(\frac{2}{2n+1}\right)-f(0)\right|$$

$$=\left|\sin\pi-\frac{2}{3}\sin\left(\frac{3\pi}{2}\right)\right|+\left|\frac{2}{3}\sin\left(\frac{3\pi}{2}\right)-\frac{2}{5}\sin\left(\frac{5\pi}{2}\right)\right|+\ldots\ldots\ldots$$

$$\ldots..+\left|\frac{2}{2n+1}\sin\left(\frac{(2n+1)\pi}{2}\right)-0\right|$$

$$=\frac{2}{3}+\left(\frac{2}{3}+\frac{2}{5}\right)+\left(\frac{2}{5}+\frac{2}{7}\right)+\ldots\ldots\ldots..+\left(\frac{2}{2n-1}+\frac{2}{2n+1}\right)+\frac{2}{2n+1}$$

$$=\left(2\left(\frac{2}{3}\right)+2\left(\frac{2}{5}\right)+2\left(\frac{2}{7}\right)+\ldots\ldots\ldots..+2\left(\frac{2}{2n+1}\right)\right)$$

$$=4\left(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\ldots\ldots..+\frac{1}{2n+1}\right)$$

Since the infinite series $\dfrac{1}{3}+\dfrac{1}{5}+\dfrac{1}{7}+\ldots\ldots..$ is divergent, therefore its partial sums sequence $\{S_n\}$, where $S_n=\dfrac{1}{3}+\dfrac{1}{5}+\dfrac{1}{7}+\ldots\ldots+\dfrac{1}{2n+1}$, is not bounded above.

Thus $\sum\left|f(x_i)-f(x_{i-1})\right|$ can be made arbitrarily large by taking n sufficiently large.

$\Rightarrow V(f;0,1)\to\infty$ and so f is not of bounded variation. ❑

Remarks

A function of bounded variation is not necessarily continuous.

e.g. the step-function $f(x)=[x]$, where $[x]$ denotes the greatest integer not greater than x, is a function of bounded variation on $[0,2]$ but is not continuous.

Theorem

If the derivative of the function f exists and is bounded on $[a,b]$, then f is of bounded variation on $[a,b]$.

Proof

$\because$ f' is bounded on $[a,b]$

$\therefore$ $\exists$ k such that $|f'(x)|\le k$ $\forall$ $x\in[a,b]$.

Let P be any partition of the interval $[a,b]$ then

$$\sum\left|f(x_i)-f(x_{i-1})\right|=\sum\left|x_i-x_{i-1}\right|f'(c)\ ,\quad c\in[a,b]\quad \text{(by M.V.T)}$$

$$\le k\left|b-a\right|$$

$\Rightarrow V(f;a,b)$ is finite. $\Rightarrow$ f is of bounded variation. ❑

Note

Boundedness of f' is a sufficient condition for $V(f)$ to be finite and is not necessary.

Theorem

A function of bounded variation is necessarily bounded.

Proof

Suppose f is of bounded variation on $[a,b]$.

For any $x \in [a,b]$, consider the partition $\{a,x,b\}$, consisting of just three points then

$$|f(x)-f(a)|+|f(b)-f(x)| \le V(f;a,b)$$
$$\Rightarrow |f(x)-f(a)| \le V(f;a,b)$$

Again

$$\begin{aligned} |f(x)| &= |f(a)+f(x)-f(a)| \\ &\le |f(a)|+|f(x)-f(a)| \\ &\le |f(a)|+V(f;a,b) < \infty \end{aligned}$$

$\Rightarrow$ f is bounded on $[a,b]$. ❑

Properties of functions of bounded variation

1) The sum (difference) of two functions of bounded variation is also of bounded variation.

Proof

Let f and g be two functions of bounded variation on $[a,b]$. Then for any partition P of $[a,b]$ we have

$$\begin{aligned} \sum\left|(f+g)(x_i)-(f+g)(x_{i-1})\right| &= \sum\left|\{f(x_i)+g(x_i)\}-\{f(x_{i-1})+g(x_{i-1})\}\right| \\ &= \sum\left|f(x_i)-f(x_{i-1})+g(x_i)-g(x_{i-1})\right| \\ &\le \sum\left|f(x_i)-f(x_{i-1})\right|+\sum\left|g(x_i)-g(x_{i-1})\right| \\ &\le V(f;a,b)+V(g;a,b) \end{aligned}$$

$$\Rightarrow V(f+g;a,b) \le V(f;a,b)+V(g;a,b)$$

This show that the function $f+g$ is of bounded variation.

Similarly it can be shown that $f-g$ is also of bounded variation.

i.e. $V(f-g) \le V(f)+V(g)$ ❑

Note

(i) If f and g are monotonic increasing on $[a,b]$ then $(f-g)$ is of bounded variation on $[a,b]$.

(ii) If c is constant, the sums $\sum\left|f(x_i)-f(x_{i-1})\right|$ and therefore the total variation function, $V(f)$ is same for f and $f-c$.

2) The product of two functions of bounded variation is also of bounded variation.

Proof

Let f and g be two functions of bounded variation on $[a,b]$.

$\Rightarrow$ f and g are bounded and $\exists$ a number k such that

$$|f(x)| \le k \quad \& \quad |g(x)| \le k \qquad \forall \ x \in [a,b].$$

For any partition P of $[a,b]$ we have

$$\begin{aligned}
&\sum\left|(fg)(x_i)-(fg)(x_{i-1})\right| \\
&=\sum\left|f(x_i)g(x_i)-f(x_{i-1})g(x_{i-1})\right| \\
&=\sum\left|f(x_i)g(x_i)-f(x_i)g(x_{i-1})+f(x_i)g(x_{i-1})-f(x_{i-1})g(x_{i-1})\right| \\
&=\sum\left|f(x_i)\{g(x_i)-g(x_{i-1})\}+g(x_{i-1})\{f(x_i)-f(x_{i-1})\}\right| \\
&\le\sum\left|f(x_i)\right|\left|g(x_i)-g(x_{i-1})\right|+\sum\left|g(x_{i-1})\right|\left|f(x_i)-f(x_{i-1})\right| \\
&\le k\sum\left|g(x_i)-g(x_{i-1})\right|+k\sum\left|f(x_i)-f(x_{i-1})\right| \\
&\le k\,V(g)+k\,V(f)
\end{aligned}$$

$\Rightarrow$ fg is of bounded variation on $[a,b]$. ❑

Note

Theorems like the above, could not be applied to quotients of functions because the reciprocal of a function of bounded variation need not be of bounded variation.

e.g. if $f(x)\to 0$ as $x\to x_0$, then $\dfrac{1}{f(x)}$ will not be bounded and therefore can not be of bounded variation on any interval which contains x_0.

Therefore to consider quotient, we avoid functions whose values becomes arbitrarily close to zero.

3) If f is a function of bounded variation on $[a,b]$ and if $\exists$ a positive number k such that $\left|f(x)\right| \ge k \quad \forall \; x\in[a,b]$ then $\dfrac{1}{f}$ is also of bounded variation on $[a,b]$.

Proof

For any partition P of $[a,b]$, we have

$$\begin{aligned}
\sum\left|\frac{1}{f}(x_i)-\frac{1}{f}(x_{i-1})\right| &=\sum\left|\frac{1}{f(x_i)}-\frac{1}{f(x_{i-1})}\right| \\
&=\sum\left|\frac{f(x_{i-1})-f(x_i)}{f(x_i)f(x_{i-1})}\right| \\
&\le\frac{1}{k^2}\sum\left|f(x_{i-1})-f(x_i)\right| \quad \le \frac{1}{k^2}V\left(f;a,b\right)
\end{aligned}$$

$\Rightarrow$ $\dfrac{1}{f}$ is of bounded variation on $[a,b]$. ❑

4) If f is of bounded variation on $[a,b]$, then it is also of bounded variation on $[a,c]$ and $[c,b]$, where c is a point of $[a,b]$, and conversely. Also

$$V(f;a,b)=V(f;a,c)+V(f;c,b).$$

Proof

a) Let, first, f be of bounded variation on $[a,b]$.

Take $P_1=\{a=x_0,x_1,.....,x_m=c\}$ & $P_2=\{c=y_0,y_1,.....,y_n=b\}$ any two partitions of $[a,c]$ and $[c,b]$ respectively.

Evidently, $P=P_1\cup P_2=\{a=x_0,....,x_m,y_0,....,y_n=b\}$ is a partition of $[a,b]$.

We have

$$\left\{\sum_{i=1}^{m}\left|f(x_i)-f(x_{i-1})\right|+\sum_{i=1}^{n}\left|f(y_i)-f(y_{i-1})\right|\right\} \le V\left(f;a,b\right)$$

$$\Rightarrow \sum_{i=1}^{m} \left| f(x_i) - f(x_{i-1}) \right| \leq V\left(f;a,b\right)$$

and $$\sum_{i=1}^{n} \left| f(y_i) - f(y_{i-1}) \right| \leq V\left(f;a,b\right)$$

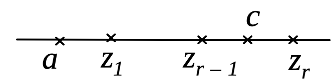
$\Rightarrow$ f is of bounded variation on $[a,c]$ and $[c,b]$ both.

b) Let, now, f be of bounded variation on $[a,c]$ and $[c,b]$ both.

Let $P = \{a = z_0, z_1, \ldots\ldots, z_n = b\}$ be a partition of $[a,b]$.

If it does not contain the point c, let us consider the partition $P^* = P \cup \{c\}$

Let $c \in \left[z_{r-1}, z_r\right]$ i.e. $z_{r-1} \leq c \leq z_r$, $r < n$



Then

$$\sum_{i=1}^{n} \left| f(z_i) - f(z_{i-1}) \right| = \sum_{i=1}^{r-1} \left| f(z_i) - f(z_{i-1}) \right| + \left| f(z_r) - f(z_{r-1}) \right| + \sum_{i=r+1}^{n} \left| f(z_i) - f(z_{i-1}) \right|$$

$$\leq \sum_{i=1}^{r-1} \left| f(z_i) - f(z_{i-1}) \right| + \left| f(c) - f(z_{r-1}) \right|$$

$$+ \left| f(z_r) - f(c) \right| + \sum_{i=r+1}^{n} \left| f(z_i) - f(z_{i-1}) \right|$$

$$\leq V\left(f;a,c\right) + V\left(f;c,b\right)$$

$\Rightarrow$ f is of bounded variation on $[a,b]$ if it is of bounded variation on $\left[a,c\right]$ & $[c,b]$ both, then

$$V\left(f;a,b\right) \leq V\left(f;a,c\right) + V\left(f;c,b\right) \quad \ldots\ldots\ldots\ldots (i)$$

Now let $\varepsilon > 0$ be any arbitrary number.

Since $V\left(f;a,c\right)$ and $V\left(f;c,b\right)$ are the total variation of f on $\left[a,c\right]$ & $[c,b]$ respectively therefore $\exists$ partition $P_1 = \{a = x_0, x_1, x_2, \ldots\ldots, x_m = c\}$ and $P_2 = \{c = y_0, y_1, y_3, \ldots\ldots, y_n = b\}$ of $\left[a,c\right]$ & $[c,b]$ respectively such that

$$\sum_{i=1}^{m} \left| f(x_i) - f(x_{i-1}) \right| > V\left(f;a,c\right) - \frac{\varepsilon}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

& $$\sum_{i=1}^{n} \left| f(y_i) - f(y_{i-1}) \right| > V\left(f;c,b\right) - \frac{\varepsilon}{2} \quad \ldots\ldots\ldots\ldots\ldots (iii)$$

Adding (ii) and (iii) we get

$$\sum_{i=1}^{m} \left| f(x_i) - f(x_{i-1}) \right| + \sum_{i=1}^{n} \left| f(y_i) - f(y_{i-1}) \right| > V\left(f;a,c\right) + V\left(f;c,b\right) - \varepsilon$$

$$\Rightarrow V\left(f;a,b\right) > V\left(f;a,c\right) + V\left(f;c,b\right) - \varepsilon$$

But ε is arbitrary positive number therefore we get

$$V\left(f;a,b\right) \geq V\left(f;a,c\right) + V\left(f;c,b\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (iv)$$

From (i) and (iv), we get

$$V\left(f;a,b\right) = V\left(f;a,c\right) + V\left(f;c,b\right)$$ ❑

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Variation Function

Let f be a function of bounded variation on $[a,b]$ and x is a point of $[a,b]$. Then the total variation of f is $V\left(f;a,x\right)$ on $[a,x]$, which is clearly a function of x, is called the *total variation function* or simply the *variation function* of f and is denoted by $V_f(x)$, and when there is no scope for confusion, it is simply written as $V(x)$.

Thus $V_f(x)=V\left(f;a,x\right) \quad ; \quad (a\leq x\leq b)$

If x_1,x_2 are two points of the interval $[a,b]$ such that $x_2>x_1$, then

$$\begin{aligned} 0\leq\left|f(x_2)-f(x_1)\right| &\leq V\left(f;x_1,x_2\right)\\ &=V\left(f;a,x_1\right)-V\left(f;a,x_2\right)\\ &=V_f(x_2)-V_f(x_1)\end{aligned}$$

$$\Rightarrow V_f(x_2)\geq V_f(x_1)$$

implies that the variation function is monotonically increasing function on $[a,b]$.

CHARACTERIZATION OF FUNCTIONS OF BOUNDED VARIATION

Theorem

A function of bounded variation is expressible as the difference of two monotonically increasing function.

Proof

We have

$$\begin{aligned} f(x)&=\frac{1}{2}\left(V(x)+f(x)\right)-\frac{1}{2}\left(V(x)-f(x)\right)\\ &=G(x)-H(x) \qquad \text{(say)}\end{aligned}$$

We shall prove that these two functions $G(x)$ and $H(x)$ are monotonically increasing on $[a,b]$.

Now, if $x_2>x_1$, we have

$$\begin{aligned} G(x_2)-G(x_1)&=\frac{1}{2}\left[V(x_2)-V(x_1)+f(x_2)-f(x_1)\right]\\ &=\frac{1}{2}\left[V(f;x_1,x_2)-\left(f(x_1)-f(x_2)\right)\right]\end{aligned}$$

Since $V(f;x_1,x_2)\geq f(x_1)-f(x_2)$

$$\Rightarrow G(x_2)-G(x_1)\geq 0 \qquad \text{i.e.} \quad G(x_2)\geq G(x_1)$$

so that the function $G(x)$ is monotonically increasing on $[a,b]$.

Again, we have

$$\begin{aligned} H(x_2)-H(x_1)&=\frac{1}{2}\left[\left(V(x_2)-V(x_1)\right)-\left(f(x_2)-f(x_1)\right)\right]\\ &=\frac{1}{2}\left[V(f;x_1,x_2)-\left(f(x_2)-f(x_1)\right)\right]\end{aligned}$$

so that as before

$$H(x_2)-H(x_1)\geq 0 \quad \text{i.e.} \; H(x_2)\geq H(x_1).$$

i.e. $H(x)$ is also monotonically increasing function.

Hence the result. ❑

Note

A function $f(x)$ is of bounded variation over the interval $[a,b]$ iff it can be expressed as the difference of two monotonically functions.

Theorem

Let f be of bounded variation on $[a,b]$. Let V be defined on $[a,b]$ as follows:

$$V(x)=V_f(x)=V\left(f;a,x\right) \quad \text{if} \quad a<x\le b\ ,\ V(a)=0 .$$

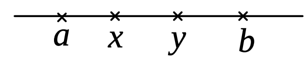
Then

i) V is an increasing function on $[a,b]$.

ii) $\left(V-f\right)$ is an increasing function on $[a,b]$.

Proof

If $a<x<y\le b$, we can write



$$V(f;a,y)=V(f;a,x)-V(f;x,y)$$

$$\Rightarrow V(y)-V(x)=V(f;x,y)$$

$$\because\ V(f;x,y)\ \ge 0$$

$$\therefore\ V(y)-V(x)\ \ge 0 \quad \Rightarrow V(x)\le V(y) \quad \text{and } (i) \text{ holds.}$$

To prove (*ii*), let $D(x)=V(x)-f(x)$ if $x\in[a,b]$.

Then, if $a\le x<y\le b$, we have

$$D(y)-D(x)=\left[V(y)-V(x)\right]-\left[f(y)-f(x)\right]$$

$$=V(f;x,y)-\left[f(y)-f(x)\right]$$

But from the definition of $V(f;x,y)$, it follows that

$$f(y)-f(x)\ \le\ V(f;x,y)$$

This means that $D(y)-D(x)\ \ge\ 0$ and (*ii*) holds. ❑

Theorem

If c be any point of $[a,b]$, then $V(x)$ is continuous at c if and only if $f(x)$ is continuous at c.

i.e. A point of continuity of $f(x)$ is also a point of continuity of $V(x)$ and conversely.

Proof

Firstly suppose that $V(x)$ is continuous at c.

Let $\varepsilon>0$ be given, then $\exists\ \delta>0$ such that

$$\left|V(x)-V(c)\right|<\varepsilon \quad \text{for} \quad \left|x-c\right|<\delta \quad \ldots\ldots\ldots\ldots (i)$$

Also, we have

$$\left|f(x)-f(c)\right|\le V(x)-V(c) \quad \text{if} \quad x>c \quad \ldots\ldots\ldots\ldots\ldots (ii)$$

And

$$\left|f(x)-f(c)\right|\le V(c)-V(x) \quad \text{if} \quad x<c \quad \ldots\ldots\ldots\ldots\ldots (iii)$$

From (*i*), (*ii*) and (*iii*), we deduce that

$$\left|f(x)-f(c)\right|\le\left|V(x)-V(c)\right|<\varepsilon \quad \text{for} \quad \left|x-c\right|<\delta$$

Which shows that $f(x)$ is continuous at c.

Now suppose that c is a point of continuity of $f(x)$ and let $\varepsilon>0$ be given, then $\exists\ \delta>0$ such that

$$\left|f(x)-f(c)\right|<\frac{\varepsilon}{2} \quad \text{for} \quad \left|x-c\right|<\delta$$

Also $\exists$ a partition $P=\left\{c=y_0,y_1,\ldots\ldots,y_{q-1},y_q,\ldots\ldots,y_n=b\right\}$ of $[c,b]$ such that

$$\sum_{q=1}^{n}\left|f(y_q)-f(y_{q-1})\right|>V(f;c,b)-\frac{1}{2}\varepsilon \quad \ldots\ldots\ldots\ldots\ldots (iv)$$

Since as a result of introducing addition points to the partition P, the corresponding sum of the moduli of the differences of the function values at end points will not be decreased, therefore we may assume that

$$0 < y_1 - c < \delta$$

so that $\left| f(y_1) - f(c) \right| < \dfrac{\varepsilon}{2}$ ……………. (v)

Thus (iv) becomes

$$V(f;c,b) - \frac{1}{2}\varepsilon < \frac{1}{2}\varepsilon + \sum_{q=2}^{n} \left| f(y_q) - f(y_{q-1}) \right| < \frac{1}{2}\varepsilon + V(f;y_1,b)$$

$$\Rightarrow V(f;c,b) - V(f;y_1,b) < \varepsilon$$

$$\Rightarrow V(y_1) - V(c) < \varepsilon$$

Thus for $0 < y_1 - c < \delta$, we have $0 < V(y_1) - V(c) < \varepsilon$

$$\therefore \lim_{x\to c+0} V(x) = V(c)$$

Similarly, we can have

$$\lim_{x\to c-0} V(x) = V(c)$$

Which shows that $V(x)$ is continuous at c. ❑

Note

$V(x)$ is continuous in $[a,b]$ iff $f(x)$ is continuous in $[a,b]$.

Corollary

A function f is of bounded variation on $[a,b]$ iff there is a bounded increasing function g on $[a,b]$ such that for any two points x' and x'' in $[a,b]$, $x' < x''$, we have

$$\left| f(x'') - f(x') \right| \le g(x'') - g(x')$$

Moreover, if g is continuous at x', so is f.

Proof

Take $g(x) = \begin{cases} V_a^x & , \ a < x \le b \\ 0 & , \ x = a \end{cases}$

Then g is increasing and bounded on $[a,b]$.

Also, $\left| f(x') - f(x'') \right| \le V_{x'}^{x''}(f) = g(x'') - g(x')$

Which also yields that if g is continuous at x', so is f. ❑

Question

Show that the function f defined by

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & ; \ x \ne 0 \\ 0 & ; \ x = 0 \end{cases}$$

is of bounded variation on $[0,1]$.

Solution

f is differentiable on $[0,1]$ and $f'(x) = 2x\sin\dfrac{1}{x} - \sin x$ for $0 \le x \le 1$.

Also

$$\left| f'(x) \right| \le \left| 2x\sin\frac{1}{x} \right| + \left| \sin x \right| \le 2 + 1 = 3$$

i.e. $f'(x)$ is bounded on $[0,1]$

Hence f is of bounded variation on $[0,1]$. ❑

Question

Show that $g(x)=\begin{cases} x\cos\dfrac{\pi x}{2} & , \ 0<x\le 1 \\ 0 & , \ x=0 \end{cases}$ is not of bounded variation on $[0,1]$

Solution

Let $P=\left\{0,\dfrac{1}{2n},\dfrac{1}{2n-1},.....,\dfrac{1}{3},\dfrac{1}{2},1\right\}$ be a partition of $[0,1]$.

Then

$$\sum\left|f(x_i)-f(x_{i-1})\right|$$
$$=\left|f(1)-f\left(\frac{1}{2}\right)\right|+\left|f\left(\frac{1}{2}\right)-f\left(\frac{1}{3}\right)\right|+\left|f\left(\frac{1}{3}\right)-f\left(\frac{1}{4}\right)\right|+......+\left|f\left(\frac{1}{2n}\right)-f(0)\right|$$
$$=\left|\cos\frac{\pi}{2}-\frac{1}{2}\cos\frac{\pi}{4}\right|+\left|\frac{1}{2}\cos\frac{\pi}{4}-\frac{1}{3}\cos\frac{\pi}{6}\right|+\left|\frac{1}{3}\cos\frac{\pi}{6}-\frac{1}{4}\cos\frac{\pi}{8}\right|+.....+\left|\frac{1}{2n}\cos\frac{\pi}{4n}-0\right|$$
$$=2\left(\frac{1}{2}\cos\frac{\pi}{4}\right)+2\left(\frac{1}{3}\cos\frac{\pi}{6}\right)+2\left(\frac{1}{4}\cos\frac{\pi}{8}\right)+.....+2\left(\frac{1}{2n}\cos\frac{\pi}{4n}\right)$$
$$=2\left(\frac{1}{2}\cos\frac{\pi}{4}+\frac{1}{3}\cos\frac{\pi}{6}+\frac{1}{4}\cos\frac{\pi}{8}+......+\frac{1}{2n}\cos\frac{\pi}{4n}\right)$$

which is not bounded.

Hence $f(x)$ is not of bounded variation on $[0,1]$. ❑

Alternative

We have

$$\left|g(x_{k+1})-g(x_k)\right|+\left|g(x_k)-g(x_{k-1})\right|$$
$$=\left|\frac{1}{k+1}\cos\frac{(k+1)\pi}{2}-\frac{1}{k}\cos\frac{k\pi}{2}\right|+\left|\frac{1}{k}\cos\frac{k\pi}{2}-\frac{1}{k-1}\cos\frac{(k-1)\pi}{2}\right|$$
$$=\begin{cases} \dfrac{2}{k} & ; \ \text{if } k \text{ is even} \\ \dfrac{1}{k+1}+\dfrac{1}{k-1} & ; \ \text{if } k \text{ is odd} \end{cases}$$

$$\Rightarrow V_a^b(g) \le \sum_{k=1}^{n}\frac{1}{k} \le \sum_{k=1}^{\infty}\frac{1}{k}$$

$\because \displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ is divergent $\therefore V_a^b(g)$ is not finite.

Hence g is not of bounded variation. ❑